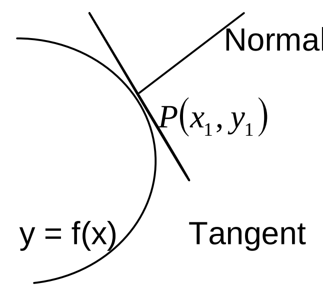
# Application of Derivative

In engineering field, we find the number of situation where we requires use of Differentiation. Here we study the following application of derivative.

1) Slope & Equation of Tangent & Normal
2) Maxima & Minima
3) Radius of Curvature

## Tangent & Normal

1) **Tangent Line**: - It is straight line which intersect curve at one point.
2) **Normal Line**: - It is straight line which is perpendicular to tangent line to the curve at point of intersection.




Geometrically, if y = f(x) be equation of any curve then $\frac{dy}{dx}$ represent slope of tangent line to curve at point of intersection.

If $P(x_1, y_1)$ be a point on the curve y = f(x) where line is tangent to curve then,

$$\text{Slope of tangent} = m = \frac{dy}{dx}/_{P(x_1,y_1)}$$

If line is Normal to curve y = f(x) at point $P(x_1, y_1)$ then,

$$\text{Slope of Normal} = m = \left(\frac{-1}{\frac{dy}{dx}}\right)_{P(x_1,y_1)}$$

Equation of Tangent is, $$(y - y_1) = \frac{dy}{dx}/_{P(x_1,y_1)}(x - x_1)$$

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Equation of Normal is, $$(y-y_1)=\left(\frac{-1}{\frac{dy}{dx}}\right)_{P(x_1,y_1)}(x-x_1)$$

**For Example:-** Find the slope of tangent & normal to the circle $x^2+y^2=25$, at the point (-3,4).

**Solution:-** Let, $x^2+y^2=25$ ------------------ (1)

Diff.w.r.t.x

$$2x+2y\frac{dy}{dx}=0 \Rightarrow \boxed{\frac{dy}{dx}=-\frac{x}{y}}$$

Slope of tangent at (-3,4) = $\left(\frac{dy}{dx}\right)_{(-3,4)}$

Put x = -3 & y = 4

$$\boxed{\text{Slope of tangent} = -\left(\frac{-3}{4}\right) = \frac{3}{4}}$$

$$\boxed{\text{Slope of normal} = \frac{-1}{Slope of\ \tan gent} = \frac{-1}{\left(\frac{3}{4}\right)} = \frac{-4}{3}}$$

**For Example:-** At What Point the curve $y=e^x$, the slope of tangent is 1.

**Solution:-** Let, $y=e^x$ ---------------------- Given

Let, $P(x_1,y_1)$ be a point on the curve where line is tangent to the curve.

$$\frac{dy}{dx}=e^x$$

But Slope of Tangent $=\frac{dy}{dx}_{(x_1,y_1)}$ = 1 --------Given

$\therefore \quad e^{x_1}=1 \Rightarrow x_1=\log_e 1$ -------Using definition of log.

$\therefore \quad \boxed{x_1=0}$ -------------- ( $\log_e 1=0$ )

From Equation of curve, $y_1=e^{x_1}=e^0=1 \Rightarrow \boxed{y_1=1}$

$\therefore$ $\boxed{\text{The point on the curve is } P(0,1)}$

**For Example :-** Find the equation of tangent & normal to the curve $y=x(2-x)$ at (2,0).

**Solution:-** Let $y=x(2-x)=2x-x^2$ -------------------Given

$$\therefore \frac{dy}{dx}=2-2x$$

$\therefore$ Slope of tangent at (2,0) = $\frac{dy}{dx}_{(2,0)}$ = $2-2(2)=$ **-2**

$\therefore$ Equation of tangent $y-y_1=-2(x-x_1)$

Put $x_1=2$, $y_1=0$

$\therefore$ $y-0=-2(x-2)$

$\therefore$ $y=-2x+4$

$\therefore$ Equation of tangent is, $\boxed{2x+y-4=0}$

$\therefore$ Also, Equation of normal is, $y-y_1=\frac{-1}{slopeof\ \tan gent}(x-x_1)$

$\therefore$ $y-0=\frac{-1}{-2}(x-2)=\frac{1}{2}(x-2)$

$\therefore$ $2y=x-2$

$\therefore$ Equation of normal is, $\boxed{x-2y-2=0}$

**Assignment-1**

1) Find slope of tangent & normal to the curve $2x^2-xy+3y^2=18$ at point (3,1).

**Solution**:-

**Answer**:-

2) At what point of the curve $y=3x-x^2$, the slope of tangent is -5.

**Solution**:-

**Answer**:-

3) At what point of the curve $y = \log(x-3)$ , the slope of tangent is 5.

**Solution**:-

**Answer**:-

4) Find the point on the curve $x^2 + y^2 = 25$ at which the tangent are parallel to X-axis.

**Solution**:-

**Answer**:-

5) Find the equation of tangent & normal to the curve $y = x^3 - 2x^2 + 4$ at x = 2.
**Solution**:-

**Answer**:-

6) Find the equation of tangent & normal to the curve $x^2 + 3xy + y^2 = 5$ at point (1,1).
**Solution:-**

**Answer**:-

**Exercise:-1**

1) Find slope of tangent & normal to the curve $13x^3+2x^2y+y^3=1$ at point (1,-2).

2) Find the point on the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where tangent are parallel to the X-axis.

3) Find the point on the curve $y=x^3-8x$ where the tangent are parallel to the line $y=4x$.

4) Find the point on the curve $y=7x-3x^2$ where the inclination of tangent is $45^0$.

5) Find the point on the curve $y=x^3-3x^2+5$ at which the normal's are parallel to Y-axis.

6) Find the equation of tangent & normal to the curve $x^2+y^2=25$ at point (3,4).

7) Find the equation of normal to the curve $3ay^2=x^2(x+a)$ at (2a,2a).

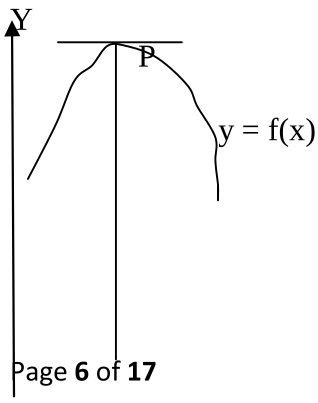
8) Find the equation of normal to the curve $y=x^2-x-6$ at the point where it crosses X-axis.

9) Show that the line $\frac{x}{a}+\frac{y}{b}=2$ touches the curve $\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m=2$ at (a,b).

10) Find equation of tangent & normal to the ellipse $\sqrt{x}-\sqrt{y}=1$ at (9,4).

## Maxima & Minima

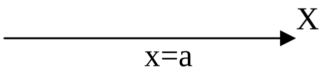
**Maxima: -** A function y = f(x) is said to have a maximum at x = a if its value at any other point in the small neighborhood around 'a' is less than the value of function at 'a'.




Let P is point of maxima on the curve where tangent is parallel to X-axis

Slope of tangent = $\frac{dy}{dx}$ = slope of X-axis

For Maxima, $\boxed{\frac{dy}{dx}=0 \text{ \& } \frac{d^2y}{dx^2} \text{ is negative at } x = a}$

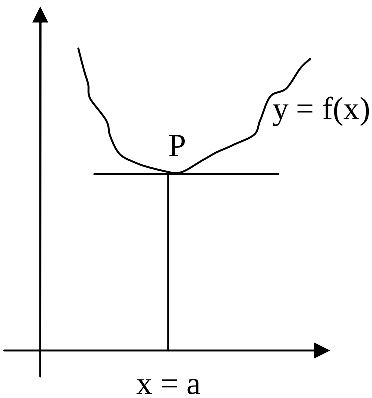



Note:-1) Maxima are hilltop point on the curve from which curve decreases in both direction.

2) The point at which $\frac{dy}{dx} = 0$ is called as stationary point.

3) Value of function y = f(x) is maximum at x = a. i.e. $\boxed{\text{Max. Value} = f(a)}$

**Minima:-** A function y = f(x) is said to have a minimum at x = a if its value at any other point in the small neighborhood around 'a' is greater than the value of function at 'a'.




Let P be a point on the curve where tangent is parallel to X-axis.

For Minima, $\boxed{\frac{dy}{dx} = 0 \ \& \ \frac{d^2y}{dx^2} \text{ is positive at } x = a}$

Note: - 1) Minima are bottommost point of cup like depression on the curve from which curve increases in both direction.

2) Value of function y = f(x) is minimum at x = a. i.e. $\boxed{\text{Min. Value} = f(a)}$

**Working Rule**:-

**Step-1** Let y = f(x) be a given function**.** Find $\frac{dy}{dx}$ & compare it to zero.

Solve equation $\frac{dy}{dx} = 0$ & find all possible values of 'x' (stationary value)

Let x = a, b, c ---- etc are stationary value

**Step-2** Find $\frac{d^2y}{dx^2}$ & check sign of it at all stationary value of 'x'**.**

Suppose x = a, then

**Case 1:-** If $\frac{d^2y}{dx^2}$ is negative at x = a then x = a is maxima & f(a) is maximum value of y = f(x).

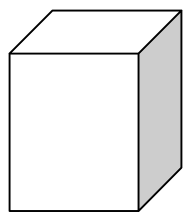
**Case 2:-** If $\frac{d^2y}{dx^2}$ is positive at x = a then x = a is minima & f(a) is minimum value of y = f(x).

**Case 3**:- If $\frac{d^2y}{dx^2} = 0$ at x = a then x = a is neither maxima nor minima.

Similarly condition are checked for x = b, c, ---------- etc

**Useful Formulae**:-

**1) Cube:-** Let 'x' be side of cube.

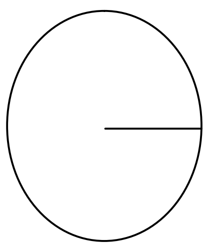


----- x ----

1) Volume = $x^3$

2) Surface Area = $6x^2$

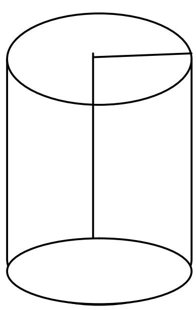
**2) Sphere**:- let 'r' is radius of sphere.



1) Volume of sphere = $\frac{4}{3}\pi r^3$

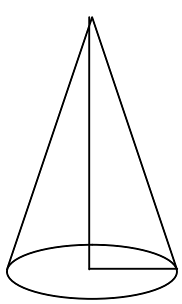
2) Surface Area = $4\pi r^2$

**3) Right Circular Cylinder: -** Let 'r' is radius of circular base & 'h' is height of cylinder**.**



Volume of Right circular cylinder = $\pi r^2 h$

**4) Right Circular Cone:-** Let 'r' is radius of circular base, 'h' is height & 'l' is slant height .



1) Area of curved surface = $\pi rl$

2) Total surface area of cone = $\pi rl + \pi r^2$

3) Volume = $\frac{1}{3}\pi r^2 h$

**For Example:-** Find maximum & minimum value of $y = 3 + 2x - x^2$.

**Solution**:- Let, $y = 3 + 2x - x^2$ ----------------- Given

Diff.w.r.t.x

$\frac{dy}{dx} = 2 - 2x$ ------------- (1)

For stationary value, $\frac{dy}{dx}=0 \Rightarrow 2-2x=0 \Rightarrow \boxed{x=1}$ ------ stationary value

There is only one value which is either maxima or minima.

Diff. (1) again w.r.t.x

$$\frac{d^2y}{dx^2}=-2$$

For x = 1, $\frac{d^2y}{dx^2}=-2$ = negative value

$\therefore$ x = 1 is maxima for given function

$\therefore$ maximum value of function = $y_{max.} = 3+2(1)-(1^2)$ = 4.

## **Assignment-2**

**1)** Find the value of 'x' for which $y=x^3-\frac{15x^2}{2}+18x$ is maximum & minimum.

**Solution:**- Let, $y=x^3-\frac{15x^2}{2}+18x$ ------------ Given

Diff.w.r.t.x

$$\frac{dy}{dx}=3x^2-\frac{15}{2}(2x)+18 = 3x^2-15x+18$$

For maximum or minimum, $\frac{dy}{dx}=0 \Rightarrow 3x^2-15x+18=0$

$\Rightarrow 3(x^2-5x+6)=0$

$\Rightarrow (x^2-5x+6)=0$

$\Rightarrow (x-2)(x-3)=0$

$\Rightarrow x=2,3$ be the stationary value.

**Answer**:-

2) Find the maximum & minimum value of $y=x^3+6x^2-15x+5$.

**Solution:-** Let, $y=x^3+6x^2-15x+5$ --------------- Given

Diff.w.r.t.x

$$\frac{dy}{dx} = 3x^2 + 6(2x) - 15 = 3x^2 + 12x - 15$$

**Answer**:-

3) Find the maximum & minimum value of $x^3 - 9x^2 + 24x$.

**Solution:-**

**Answer**:-

4) A manufacturer can sell 'x' items at a price of Rs. $(330 - x)$each. The cost of producing 'x' items is Rs. $(x^2 + 10x + 12)$.Determine number of items to be sold so that the manufacturer can make the maximum profit.

**Solution:-** Let, Selling price of an item = Rs. $(330 - x)$

$\therefore$ Selling price of 'x' item = Rs. $(330 - x).\,x$

$\Rightarrow$ Total Selling Price (S.P.) = $330x - x^2$

Also cost of Production (C.P) = Rs. $(x^2 + 10x + 12)$.

$\therefore$ Profit (P) = S.P. – C.P.

P = $(330x - x^2)$ - $\left(x^2 + 10x + 12\right)$

$\boxed{P = -2x^2 + 320x - 12}$

(Hint:- Find maxima for function 'P')

**Answer**:-

5) A slant height of cone is 4 unit. Find the dimension of the cone which has maximum volume.
**Solution:-** ( Hint :- Find maxima for Volume (v) )

**Answer**:-

6) A bullet is fired into a mud bank and penetrates $\left(120t - 3600t^2\right)$ meter in 't' second after impact. Calculate the maximum depth of penetration.

**Solution**:- Let, Depth (D) = $120t - 3600t^2$

(Hint:- find maxima for depth (D) )

**Answer**:-

**<u>Exercise:-2</u>**

1) Find maximum & minimum value of following function.

a) $y = 4 - x - x^2$   b) $y = 2x^3 - 9x^2 + 12x + 5$   c) $y = x^3 - x^2 - 5x + 13$

d) $y = \frac{4}{x} + \frac{36}{2 - x}$   e) $y = x(k - x)$   f) $y = x^4 - 6x^2 + 8x - 30$

2) A topless box of maximum capacity is to be prepared by a square sheet 90m long by removing equal squares at the corners and bending the sheet to form the box. Find the height of the box.

3) A right circular cylinder is to be made so that the sum of its radius & height is 6m. find maximum volume of cylinder.

4) The bending moment of the beam supported at the end & uniformly loaded at a distance ‘x’ from one end is given by $M = \frac{WL}{2}x - \frac{W}{2}x^2$ where W is the load of the beam per unit run. Find a point on the beam at which the bending moment is maximum.

## <u>Radius of Curvature</u>

Curvature of curve is nothing but bending of the curve at particular point of the curve. It is denoted by ‘k’.
Radius of curvature is reciprocal of curvature of curve. It is denoted by ‘ ° ’.

$$\therefore \quad \circ = \frac{1}{k}$$

If y = f(x) be any function. Let $P(x_1, y_1)$ be any point on the curve y = f(x).
Then Radius of Curvature is given by,

$$\boxed{\text{Radius of curvature } (\circ) = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \text{ at point } P(x_1, y_1).}$$

**For Example:-** Find the radius of curvature for $y = x^3 + 3x^2 + 2$ at point (1,2).

**Solution**:- Let, $y = x^3 + 3x^2 + 2$ ------------------- Given

Diff.w.r.t.x

$$\frac{dy}{dx} = 3x^2 + 6x \quad \& \quad \frac{d^2y}{dx^2} = 6x + 6$$

<u>At point (1,2)</u> :- $\frac{dy}{dx} = 3(1)^2 + 6(1) = 9$ & $\frac{d^2y}{dx^2} = 6(1) + 6 = 12$

Radius of curvature at (1,2) = $\circ_{(1,2)} = \frac{\left[1+(9)^2\right]^{\frac{3}{2}}}{12} = \frac{[82]^{\frac{3}{2}}}{12}$

$$\Rightarrow \boxed{\circ_{(1,2)} = \frac{82\sqrt{82}}{12} \text{ unit}}$$

## <u>Assignment-3</u>

**1)** Find radius of curvature of $y = x^2$ at $x = \frac{1}{2}$.

**Solution**:- Let, $y = x^2$ ------------- Given

Diff.w.r.t.x

$$\frac{dy}{dx} = 2x \quad \& \quad \frac{d^2y}{dx^2} = 2$$

At point $x = \frac{1}{2}$, $\frac{dy}{dx} = 2\left(\frac{1}{2}\right) = 1$ & $\frac{d^2y}{dx^2} = 2$

**Answer**:-

**2)** Find the radius of curvature of $y^2 = 4ax$ at x = a.

**Solution**:- Let, $y^2 = 4ax$ -------------- Given (Implicit function)

Diff.w.r.t.x

$$2y\frac{dy}{dx} = 4a \quad \Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2\sqrt{ax}} = \sqrt{\frac{a}{x}}}$$

Diff. again w.r.t.x

$$\boxed{\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{2a}{4ax}\left(\frac{\sqrt{a}}{\sqrt{x}}\right) = -\frac{\sqrt{a}}{2x\sqrt{x}}}$$

**Answer**:-

3) Find the radius of curvature for $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (0,b).

**Solution:-**

**Answer**:-

4) A beam is bent in the form of curve $y = 2\sin x - \sin 2x$. Find the radius of curvature at $x = \frac{®}{2}$.

**Solution:-**

**Answer**:-

5) Find the radius of curvature of curve $y = a\left(1-\cos ¯\right)$, $x = a\left(¯ - \sin ¯\right)$ at $¯ = ®$.

**Solution:-**

**Answer**:-

**6)** Show that the radius of curvature at any point $(x_1, y_1)$ on the curve $x.y = c^2$ is given by $\frac{r^3}{2c^2}$ where $r = \sqrt{{x_1}^2 + {y_1}^2}$ .

**Solution:-**

**Answer**:-

| **Exercise:-3** |
|---|

**1)** Find the radius of curvature of following curve at given point.

a) $y^2 = 4x$ at $\left(2, 2\sqrt{2}\right)$ b) $y = e^x$ at (0,1) 3) $y = x^3$ at (2,8)

d) $y = \log(\sin x)$ at $x = \frac{\text{®}}{2}$ e) $x = a\cos^{3-}$ , $y = a\sin^{3-}$ at $- = \frac{\text{®}}{4}$

**2)** A Telegraph wire hangs in the form of curve $y = a\log\left(\sec\frac{x}{a}\right)$ where 'a' is constant. show that the radius of curvature at any point is $a\sec\left(\frac{x}{a}\right)$.

3) Show that, radius of curvature of circle $x^2 + y^2 = r^2$ is constant at any point.