# Definite Integral

**For** $\int f(x)dx = F(x)$

Then the definite integral of between the limits x = a and x = b is defined as

$$\int_a^b f(x)dx = [F(x)]_{x=a}^{x=b} = F(b) - F(a)$$

**Some Properties of definite integral**

**Property 1:** $\int_a^b f(x)dx = \int_a^b f(t)dt$

**Property 2:** $\int_a^b f(x)dx = -\int_b^a f(x)dx$

**Property 3:** for a<c<b

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

**Property 4:** $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

**Property 5:** $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

**Property 6:** $\int_{0}^{2a} f(x)dx$ = $\int_{0}^{a} f(x)x + \int_{0}^{a} f(2a-x)dx$

**Property 7:** $\int_{-a}^{a} f(x)dx$ = $\int_{0}^{a}[f(x)+f(-x)]dx$ = $2\int_{0}^{a} f(x)dx$

**Ex. Evaluate** $\int_{0}^{\infty} e^{-x}dx$

**Let** I = $\int_{0}^{\infty} e^{-x}dx$ = $\left[\frac{e^{-x}}{-1}\right]_{o}^{\infty}$ = $-\left[e^{-\infty}-e^{0}\right]$

= $\left[-e^{-\infty}-e^{0}\right]$ = $-0+1$

I = 1

**Ex. Evaluate** $\int_{2}^{11} \frac{1}{2x+11}dx$

**Let** I = $\int_{2}^{11} \frac{1}{2x+11}dx$

= $\frac{[\log(2x+11)]_{2}^{11}}{\frac{d}{dx}(2x+11)}$ ...... $\int_{a}^{b} \frac{1}{f(x)}dx$ = $\frac{[\log f(x)]_{a}^{b}}{\frac{d}{dx}f(x)}$

= $\frac{\log(2\times 11+11)-\log(2\times 2+11)}{2}$

= $\dfrac{\log(2\times 11+11)-\log(2\times 2+11)}{2}$

= $\dfrac{1}{2}[\log(33)-\log(15)]$

= $\dfrac{1}{2}\log\left[\dfrac{33}{15}\right]$ **..... Ans.**

**Ex. Evaluate** $\displaystyle\int_0^1 \frac{dx}{4-x^2}$

**Let** I = $\displaystyle\int_0^1 \frac{dx}{4-x^2}$ = $\displaystyle\int_0^1 \frac{dx}{(2)^2-x^2}$

Using $\displaystyle\int_p^q \frac{dx}{a^2-x^2}$ = $\left[\dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right|\right]_p^q$ , we get

I = $\dfrac{1}{2\times 2}\left[\log\left|\dfrac{2+x}{2-x}\right|\right]_{x=0}^{x=1}$

= $\dfrac{1}{4}\left[\log\left|\dfrac{2+1}{2-1}\right|-\log\left|\dfrac{2+0}{2-0}\right|\right]$

= $\dfrac{1}{4}[\log 3-\log(1)]$

= $\dfrac{1}{4}\left[\log\left|\dfrac{3}{1}\right|\right]$ **....** $\log m-\log n=\log\dfrac{m}{n}$

= $\dfrac{1}{4}(\log 3)$ **.... Ans.**

**Ex. Evaluate** $\int_0^2 \frac{5x+2}{x^2+4}dx$

**Let** I = $\int_0^2 \frac{5x+2}{x^2+4}dx$

= $\int_0^2 \frac{5x}{x^2+4}dx + \int_0^2 \frac{2}{x^2+4}dx$

= $\frac{5}{2}\int_0^2 \frac{2x}{x^2+4}dx + 2\int_0^2 \frac{dx}{x^2+2^2}$ **....** Adjustment of 2

= $\frac{5}{2}\left[\log\left|x^2+4\right|\right]_0^2 + 2\cdot\frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$

**...** $\int_a^b \frac{f'(x)}{f(x)}dx$ = $\left[\log f(x)\right]_a^b$ and $\int_p^q \frac{1}{x^2+a^2}$ = $\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]_p^q$

**?**

**HW Ex. 1.** $\int_{0}^{2} \frac{x+1}{\sqrt{x}} dx$

**HW Ex. 2.** $\int_{0}^{\Pi/2} \sin 5x \cdot \cos 3x dx$

**HW Ex. 3.** $\int_0^4 \frac{dx}{\sqrt{4x - x^2}}$

**Ex. Evaluate** $\int_{1}^{3} \frac{dx}{\sqrt{x^2 - 6x + 13}}$

**Let** I = $\int_{1}^{3} \frac{dx}{\sqrt{x^2 - 6x + 13}}$

To make perfect square

Write $x^2 - 6x + 13$ = $x^2 - 6x + 9 + 4$ = $(x-3)^2 + 2^2$

I = $\int_{1}^{3} \frac{dx}{\sqrt{(x-3)^2 + 2^2}}$

Using $\int_{a}^{b} \frac{dx}{\sqrt{x^2 + a^2}}$ = $\log\left|x + \sqrt{x^2 + a^2}\right|_a^b$

= $\left\{\log\left|(x-3) + \sqrt{(x-3)^2 + 2^2}\right|\right\}_1^3$

**?**

**H.W.Ex. 1.** $\int_0^2 \frac{dx}{\sqrt{3+2x-x^2}}$

**Problem on property of definite integration**

**(i)** $\int_a^b f(x)dx$ **=** $\int_a^b f(a+b-x)dx$ **(ii)** $\int_0^a f(x)dx$ **=** $\int_0^a f(a-x)dx$

**Ex. Evaluate** $\int_4^5 \frac{\sqrt{x-4}}{\sqrt{x-4}+\sqrt{5-x}}dx$

**Let** I = $\int_4^5 \frac{\sqrt{x-4}}{\sqrt{x-4}+\sqrt{5-x}}dx$ ... (1)

By using property

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$ Here a = 4, b = 5

Equation (1) becomes

$$I = \int_4^5 \frac{\sqrt{(4+5-x)-4}}{\sqrt{(4+5-x)-4}+\sqrt{5-(4+5-x)}}dx$$

$$= \int_4^5 \frac{\sqrt{4+5-x-4}}{\sqrt{4+5-x-4}+\sqrt{5-4-5+x}}dx$$

$$= \int_4^5 \frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{-4+x}}dx \qquad = \int_4^5 \frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{-4+x}}dx \qquad \dots (2)$$

By adding equations (1) and (2) we get

$$I + I = \int_4^5 \frac{\sqrt{x-4}}{\sqrt{x-4}+\sqrt{5-x}}dx + \int_4^5 \frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{-4+x}}dx$$

$$2I = \int_4^5 \frac{\sqrt{x-4}+\sqrt{5-x}}{\sqrt{x-4}+\sqrt{5-x}}dx$$

$$2I = \int_4^5 dx = [x]_4^5 = 5-4 = 1$$

$$I = \frac{1}{2}$$ … Ans.

**HW Ex. 1.** $\int_3^5 \frac{\sqrt{8-x}}{\sqrt{x}+\sqrt{8-x}}dx$

**HW Ex. 2.** $\int_{1}^{2} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}\,dx$

**HW Ex. 3.** $\int_{2}^{7} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\,dx$

**HW Ex. 4.** $\int_{2}^{7} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

**HW Ex. 5.** $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\,dx$

**HW Ex. 6.** $\displaystyle\int_0^{\pi/2} \frac{\sin x}{(\sin x+\cos x)^2}\,dx$

**Ex. Evaluate** $\int_0^{®} x \cdot \sin x dx$

**Let** I = $\int_0^{®} x \cdot \sin x dx$

$\int_0^a f(x) dx$ = $\int_0^a f(a-x) dx$ .... Using Property

I = $\int_0^{®} (® - x) \cdot \sin(® - x) dx$

I = $\int_0^{®} (® - x) \cdot \sin x dx$ .... $\sin(® - x) = \sin x$

I = $\int_0^{®} (® \sin x - x \sin x - x) dx$

I = $\int_0^{®} ® \sin x dx - \int_0^{®} x \sin x dx$ -

I = $\int_0^{®} ® \sin x dx - \text{I}$

$\therefore$ I+ I = $® \int_0^{®} \sin x dx - \text{I}$ .... As $\Pi$ is constant

2I = $® [-\cos x]_0^{®}$ .... $\int_a^b \sin x dx = [-\cos x]_a^b$

2I = $-® [\cos ® - \cos x]$ = $-® [-1-1]$

$(\because \cos ® = -1 and \cos 0 = 1)$

2I = $-® (-2)$ = $2®$

I = ®  … Ans.

**Ex. Evaluate** $\int_{0}^{®} \frac{x}{1+\sin x} dx$

**Let** I = $\int_{0}^{®} \frac{x}{1+\sin x} dx$ = $\int_{0}^{®} \frac{® - x}{1+\sin(® - x)} dx$

= $\int_{0}^{®} \frac{® - x}{1+\sin x} dx$ = $\int_{0}^{®} \frac{®}{1+\sin x} dx$ - $\int_{0}^{®} \frac{x}{1+\sin x} dx$

= $\int_{0}^{®} \frac{®}{1+\sin ®} - I$ …. $I = \int_{0}^{®} \frac{x}{1+\sin x} dx$

2I = $® \int_{0}^{®} \frac{1}{1+\sin x} dx$

Now multiplying and dividing by $1 - \sin dx$

2I = $® \int_{0}^{®} \frac{1}{1+\sin x} dx \times \frac{1-\sin x}{1-\sin x} dx$

**?**

**Ex. Evaluate** $\int_0^1 x^2\sqrt{1-x}\,dx$

**Let** I = $\int_0^1 x^2\sqrt{1-x}\,dx$ = $\int_0^1 (1-x)^2\sqrt{1-(1-x)}\,dx$ .... $\int_0^a f(x)dx$ = $\int_0^a f(a-x)dx$

= $\int_0^1 \left(1-2x-x^2\right)\sqrt{1-1+x}\,dx$

**?**

**Ex. Evaluate** $\int_0^4 \frac{8x}{x^2+4}dx$

**Let** I = $\int_0^4 \frac{8x}{x^2+4}dx$ = $\int_0^4 \frac{4 \cdot 2x}{x^2+4}dx$

$f'(x)$

$\downarrow$

= $4\int_0^4 \frac{2x}{x^2+4}dx$ .... $\int_a^b \frac{f'(x)}{f(x)}dx = [\log f(x)]_a^b$

$\uparrow$

$f(x)$

**?**

**Ex. Evaluate** $\int_{0}^{\text{®}/2} \frac{\cos x}{1+\sin x^2} dx$

**Let** I = $\int_{0}^{\text{®}/2} \frac{\cos x}{1+\sin x^2} dx$

Put $\sin x = t$ differentiating w.r.t.x

$\therefore \quad \frac{d}{dx}\sin x = \frac{d}{dx}(t)$

$\therefore \quad \cos x = \frac{dt}{dx}$

$\therefore \quad \cos x dx = dt$

For $\sin x = t$

put x = 0 $\quad\therefore\quad \sin x = t$, i.e. $t = 0$

put x = $\frac{®}{2}$ $\quad\therefore\quad \sin\frac{®}{2} = t$, i.e. $t = 1$

| | | |
|---|---|---|
| $\therefore x$ | 0 | $®/2$ |
| t | 0 | 1 |

$\therefore \quad I = \int_0^1 \frac{dt}{1+t^2}$

$= \left[\tan^{-1} t\right]_0^1 \qquad .... \int_a^b \frac{1}{1+x^2}dx = \left[\tan^{-1} x\right]_a^b$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{®}{4} - 0 \qquad .... \tan^{-1}(1) = \frac{®}{4}, \tan^{-1}(0) = 0$

$I = \frac{®}{4}$ ....**Ans.**

**H.W.Ex. 1.** $\int_0^{®/2} \frac{\cos x dx}{4 - \sin^2 x}$

**Ex. Evaluate** $\int_0^{\pi/4} \log(1+\tan x)dx$

Using $\int_0^a f(x)dx$ **=** $\int_0^a f(a-x)dx$

$$I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx \qquad \text{.... (1)}$$

Now $\tan\left(\frac{®}{4}-x\right) = \dfrac{\tan\frac{®}{4}-\tan x}{1+\tan\frac{®}{4}\cdot\tan x}$ .... $\tan(A-B) = \dfrac{\tan A-\tan B}{1+\tan A\cdot\tan B}$

$= \dfrac{1-\tan x}{1+\tan x}$ .... $As\left[\tan\left(\frac{®}{4}\right)=1\right]$

$\therefore$ Equation (1) becomes

I $= \int_0^{®/4}\log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx = \int_0^{®/4}\log\left[\frac{1-\tan x+1-\tan x}{1+\tan x}\right]dx$

I $= \int_0^{®/4}\log\left[\frac{2}{1+\tan x}\right]dx = \int_0^{®/4}[\log 2-\log(1+\tan x)]dx$

I $= \int_0^{®/4}\log 2dx - \int_0^{®/4}\log(1+\tan x)dx = \log 2\int_0^{®/4}dx - I$

2I $= \log 2\int_0^{®/4}dx$

2I $= \log 2[x]_0^{®/4} = \log 2\left[\frac{®}{4}-0\right]$

I $= \log 2\left[\frac{®}{8}\right]$ ....**Ans.**

**H.W.Ex.** **1.** $\int\frac{(x^2+1)^2}{x}dx$ **2.** $\int x\cdot e^{2x}dx$ **3.** $\int_0^2\frac{2x}{x^2+4}dx$

**4.** $\int\frac{dx}{x^2+x}dx$ **5.** $\int\frac{dx}{x+\sqrt{x}}dx$ **6.** $\int\frac{x^2\cdot\tan^{-1}x}{1+x^2}dx$

**7.** $\int_{0}^{\text{®}/2} \frac{dx}{4+5\cos x} dx$ **8.** $\int_{0}^{\text{®}/4} \log(1+\tan x) dx$ **9.** $\int_{2}^{11} \frac{\sqrt{13-x}}{\sqrt{x}+\sqrt{13-x}} dx$

**10.** $\int \tan^{-1} x dx$ **11.** $\int_{0}^{\text{®}/2} \frac{\cos x}{1+\sin^{2} x} dx$